Philosophy 324A
Philosophy of Logic
2016

Note One

A Brief Overview of Classical First-Order Logic

Classical logic is made up of *classical propositional logic* (CPL) and *classical quantification theory* (CQT), also known as the classical predicate calculus. In this course, many of the basic philosophical questions arise at the propositional level (we'll deal with some exceptions later, as they arise). But it will be necessary to refresh our memories about classical quantificational logic as well.

We'll being with CPL, and follow with CQT.

Classical Propositional Logic (CPL)

A *logistic system* (Alonzo Church 1903-1995) is an ordered quadruple ⟨Grammar, Proof Theory, Semantics, Metatheory⟩.

Grammar of CPL

The grammar of CPL is furnished by *vocabulary* and a set of *formation rules* for a formal language L.

Vocabulary:

1. p, q, r, s, p_1, …, p_n, …, s_1, …, s_n, … are *atomic* sentences of CPL.

2. ~, ∧, ∨, ⊃, ≡ are *sentence-connectives* of CPL.

3. The parentheses (, and) are *punctuators* of CPL

4. The brackets ⟨, and ⟩ are *sequence-markers* of CPL.

Formation Rules

1. If A is an atomic sentence, it is a sentence.

2. If A is a sentence, so is ~A.

3. If A, B are sentences, so are

A ∧ B
A ∨ B

$A \supset B$
$A \equiv B$

4. Nothing else is a sentence of CPL.

The formation rules recursively define all the formal *sentences* of CPL. It is also possible to define *sequences* of sentences.

5. If $A_1, \ldots, A_n$ are a finite number of sentences then $\langle A_1, \ldots, A_n \rangle$ is a *finite sequence* of those sentences, i.e., an ordered n-tuple of them.

Proof Theory of CPL

Note: There are three equally acceptable ways of laying out the proof theory of our logic. We can do so axiomatically, or we can use a natural deduction method, or we can use a tree method. For present purposes it doesn't matter what approach we take. Let us look at a version of the axiomatic approach set out in Whitehead and Russell's *Principia Mathematica,* 3 volumes. Cambridge University Press, 1910-1913.

Axiom Schemata for CPL

A 1. $(A \vee A) \supset A$
A 2. $B \supset (A \vee B)$
A 3. $(A \vee B) \supset (B \vee A)$
A 4. $(B \supset C) \supset ((A \vee B) \supset (A \vee C))$
A 5. $((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$

Two remarks about the axioms: As they occur here, the connectives are uninterpreted. Sentences lack truth values. Hence the axioms themselves are truth valueless. They are, so to speak, just "marks on paper". Also, our axioms use only two of the five connectives. Why? Because CPL is functionally complete. This means, in particular, that any set whatever of any connectives reduces to the pair $\{\vee, \supset\}$. (Also to $\{\sim, \wedge\}$ and $\{\sim, \supset\}$ and the unary Steffer stroke |.

Transformation rules

TR 1. *Uniform substitution*. If A is a theorem so too is the result of replacing any atom of A in all its occurrences by any sentence.

TR 2. *Detachment*. If A and $(A \supset B)$ are theorems so also is B.

Theorems

Th 1. Any instantiation of an axiom schema is a theorem.

Th 2. Any sentence arises from the application of a transformation rule to one or more theorems finitely many times is a theorem. (In other words, the property of being a theorem *is closed under the transformation rules*).

Note. As used here, the word "theorem" has a technical meaning. In ordinary English, a theorem is a sentence whose truth can be demonstrated. But in CPL, theorems don't have truth values. Again, they are just a special class of "marks on paper".

Deductions

D1. If $A_1, \ldots, A_n$ are sentences, then the sequence $\langle A_1, \ldots, A_n \rangle$ is a deduction of A_n from the set $\{A_1, \ldots, A_{n-1}\}$ iff A_n arises from prior lines by finite application of one or both of the transformation rules.

D2. If $\langle A_1, \ldots, A_n \rangle$ is a deduction, then A_n is *deducible from* $\{A_1, \ldots, A_{n-1}\}$, and vice versa.

Proofs

P1. A deduction $\langle A_1, \ldots, A_n \rangle$ is a *proof* of A_n from $\{A_1, \ldots, A_{n-1}\}$ iff each of $\langle A_1, \ldots, A_n \rangle$ is either an axiom or arises from axioms by finite application of the transformation rules.

P2. If $\langle A_1, \ldots, A_n \rangle$ is a proof of A_n from $\{A_1, \ldots, A_{n-1}\}$, then A_n is *provable from* $\{A_1, \ldots, A_{n-1}\}$, and vice versa.

Note: Since axioms are theorems and the transformation rules preserve theoremhood, every line of a proof is a theorem. Note also that, when A is a theorem, the one-membered sequence (A) proves it to be.

Semantics of (Model Theory)[1] *CPL*

Heads up: In logic the word "semantics" is misleading. It is a term antecedently in use by linguists to denote a theory of meaning, but in logic a semantics or semantic theory is a theory of *models*; and it is characteristic of models to exclude meanings in favour of the other factors that determine the truth values of formal sentences. John Burgess is good on this point: "To avoid confusion, one could distinguish "*formal* semantics" or model theory from "linguistic semantics" or meaning theory; but it is best to avoid "semantics" altogether". (*Philosophical Logic*, Princeton, 2009; p. 20).

Note: Here, too, we can approach the semantics or model theory for CPL in a number of ways. But the most efficient of these is the truth table method, which itself is a variation of a natural deduction approach.[2]

Truth values: The truth values of CPL are T and F.

[1] Also called "truth theory".
[2] The truth table method for checking for validity was introduced by Emil Post (1897-1954).

Atomic valuations: An atomic valuation of CPL is a function which assigns to each atomic sentence either T or F but not both.

Interpretation: An Interpretation, I of CPL is a function which, relative to a given atomic valuation, assigns a unique truth value, T or F, to each sentence of CPL An interpretation of CPL makes its truth value assignments in accordance with the following conditions.

I (*at*): A valuation that provides for any atomic sentences one or other of the truth value is assigned. In other words $\nu(A) = T$ (or F), for each A for each atomic A. (It doesn't matter which they receive, so long as each receives exactly one of the two). Note: $\nu(A) = T$ means "the truth value of A under assignment I is T" (similarly for $\nu(F)$).

I ($\vee$): For any sentence $(A \vee B)$, $\nu(A \vee B) = F$ when $\nu(A) = F$ and $\nu(B) = F$, and otherwise is T. This is summarized by the truth table for $\vee$.

A	B	$(A \vee B)$
T	T	T
T	F	T
F	T	T
F	F	F

I ($\supset$): For any sentence $(A \supset B)$, $\nu(A \supset B) = T$ except where $\nu(A) = T$ and $\nu(B) =$ F. This is summarized by the truth table for $\supset$.

A	B	$(A \supset B)$
T	T	T
T	F	F
F	T	T
F	F	T

Note: For any sentence of CPL there is a truth table for it.

Logical truths: If A is a sentence, A is a *logical truth* on I iff for every valuation on the atoms of A, $\nu(A) = T$. In other words, A is a logical truth iff it is made true by every atomic valuation. Put yet another way, A is a logical truth iff it takes T in every row of its truth table.

Tautologies: In CPL logical truths are called *tautologies*.

Sentence-validity: In some treatments, logical truths or tautologies are called *valid sentences.* These usages are freely interchangeable.

Entailment: A *entails* B on I iff every atomic valuation assigning T to A assigns T to B. Alternatively A entails B iff (A ⊃ B) is a tautology. Yet another way, A entails B iff there is no row in the truth table for A ⊃ B in which A is T and B is F.

Sequence-validity: A sequence $\langle A_1, \ldots, A_n\rangle$ is *valid* iff every atomic valuation making $A_1, \ldots, A_{n-1}$ all true also makes A_n true. Alternatively, $\langle A_1, \ldots, A_n\rangle$ is valid iff every row of the truth table for $A_1, \ldots, A_n$ that assigns T to $A_1, \ldots, A_{n-1}$ also assigns T to A_n.

Demonstrations

A valid sequence $\langle A_1, \ldots, A_n\rangle$ is a demonstration of A_n iff every line of the sequence is a tautology.

Metatheory of CPL

A big question is, "Why do we bother with proof theory?" After all, its principal concepts – axiom, theorem, deduction, proof – have no intuitive meaning there. What's the point?

Suppose we could show that for each of these uninterpreted properties of CPL's proof theory theory is a unique counterpart in its semantics? Suppose we could show that something is a theorem of the proof theory exactly when it is a valid sentence of the semantics? Suppose we could show that whenever A ⊃ B is derivale in the proof theory then A entails B in the semantics, and *vice versa*? Would this be important? It would. It would show that the semantically interpretible properties of the logic are describable and recognizable in a *purely formal* way. In fact, the answer to these (and similar) questions is Yes. The principal metatheoretical results of CPL are as follows.

> *Soundness.* CPL is sound. Every theorem is a tautology. Alternatively, every provable sentence is logically true.
>
> *Complete*. CPL is complete. Every tautology is a theorem. Alternatively, every logical truth is provable.
>
> *Syntactic Consistency*: CPL is syntactically consistent. For no A is it the case that both A and ~A are both theorems.
>
> *Semantic Consistency*. CPL is semantically consistent. For no A is it the case that A is T and ~A is T.
>
> *Syntactic Deduction Property.* A_n is deducible from $\{A_1, \ldots, A_{n-1}\}$ iff $(A_{n-1} \supset A_n)$ is derivable from $\{A_1, \ldots, A_{n-2}\}$.
>
> *Semantic Deduction Property*. A_n is entailed by $\{A_1, \ldots, A_{n-1}\}$ iff $(A_{n1} \supset A_n)$ is

entailed by $\{A_1, \ldots, A_{n-2}\}$.

Combining the Above: CPL has both deduction properties. Moreover, whenever something is deducible from something in the proof theory it is entailed by that something in the semantics; and vice versa.

Decidability. A property P is decidable iff with respect to any arbitrarily selected object it can be determined *mechanically*, *infallibly* and in *finite time* whether or not that object has P. In CPL, the following properties are decidable.

1. The property of being a *sentence*
2. The property of being a *theorem* (*logical truth*)
3. The property of being a *deduction* (*valid sequence*)
4. The relation of *deducibility* (*entailment*)
5. And, in axiomatic approaches, the property of being a *proof* (*demonstration*).

Classical Quantification Theory (CQT)

In the interest of space, we'll omit the Proof Theory of CQT

Grammar of CQT

CQT is an extension of CPL, got by enriching its vocabulary, and adjusting accordingly the rules of grammar and semantics. In the interest of space, we'll simply assume that CPL is already part of CQT, and will begin with the rules specific to it.

Vocabulary:

Names: *a, b, c,* a_1, a_2, …, a_n, ….

Predicates: F^1_1, F^1_2, … F^1_n, …, F^2_1, F^2_2, …, F^2_n …; F^n_1, F^n_2, …, F^n_n, ….

Individual Variables: *x, y, z,* x_1, x_2, …, x_n, ….

Quantifier: ∀

Note: A *singular term* of CQT is either a name or individual variable.

Formation Rules (specific to CQT)

1. $A(\alpha_1, \ldots, \alpha_n)$ is a formula when A is an n-ary predicate and $\alpha_1, \ldots, \alpha_n$ are singular terms.

2. If A is a formula and α an individual variable, then ∀α(A) is a formula (α is said to be the variable of the quantifier).

3. If A is a formula so is $\sim A$
4. If A, B are formulas, so too are

$A \wedge B$
$A \vee B$
$A \supset B$
$A \equiv B$.

Further Parts of the Grammar

a. *Scope*. If $\forall\alpha A$ is a formula, then A is the scope of $\forall$.

b. *Freedom and bondage of occurrences.* An occurrence of a variable α in a formula is *bound* in a formula iff either it is the variable of a quantifier or it occurs in the scope of a quantifier and is a variable of it. An occurrence of a variable in a formula is free iff it is not bound in that occurrence there.

c. *Freedom and bondage of variables.* A variable α is *free* or *bound* in a formula A according as A contains free or bound occurrences of α. (So a variable can be both free and bound in a formula).

d. *Substitution instances.* If A and B are formulas, then B is a substitution instance of A iff B is the formula $S^{\beta}_{\alpha}A$, where this is the formula that results from substituting the singular term α for all free occurrences of β in A.

e. *Freedom for.* Let β a variable of a formula A and α a singular term. Then β is free for α in A iff either α itself is a variable occurring free in $S^{\beta}_{\alpha}A$ or α is a name.

f. *Closed formulas* A is a closed formula iff it contains no free occurrences of a variable.

g. *Open formulas.* A is an open formula iff it is not closed.

h. *Quantifier closure.* Let A be a formula open at one or more occurrences of the variables $\alpha_1, \ldots, \alpha_n$. Then $\forall\alpha_1, \ldots, \forall\alpha_n(A)$ is the quantifier closure of A (similarly for any order of the $\forall\alpha_i$).

i. *Further facts about closure.* If A is a closed formula then A is the closure of itself and $\forall\alpha(A)$ is a closure of A, where α is a variable. If B is a closure of A and A is a closure of B, then A is a closure of B. (In other words, the property of being a closure of is transitive).

j. *The quantifier* $\exists$. $\exists\alpha(A)$ can be defined as $\sim\forall\alpha\sim(A)$.

Semantics of CQT

An *interpretation* I is a countably infinite set D of arbitrary objects called *individuals,* supplemented by two objects undefined T and F, together with some funtions assigning the vocabulary types of the grammar to elements or combinations of elements in D. D is called the *domain of discourse.*

1. *Names*: If α is a name, then I(α) is a unique object in D (intuitively, what α names in D).

2. *Predicates*: Every n-ary predicate denotes under I a set of ordered n-tuples of individuals from D. Intuitively, these are the individuals related to one another by the relation in question. These predicates are called n-place relations. If n = 1, they are n-place properties.

3. *Variables*: See below.

4. *Connectives*: As usual.

5. *Quantifiers*: See below.

Satisfaction

Under I the formulas of CQT are either *satisfied* or not *satisfied by any countably infinite sequence of individuals from D.* Something is countably infinite iff its cardinality is aleph-null, i.e., the cardinality of the natural numbers.

Individual variables. Let A be any formula in which α_1 is a variable occurring free. Let Σ be a countably long sequence of objects of D. Then α_i denotes the *i*th member of Σ, i.e., the object that occurs in *i*-th place in Σ. If Σ* is a different sequence, then α_i will denote the *i*th member of it. But the *i*th member of Σ* needn't be the same object of the *i*th object of Σ. Generalizing: the variable α_i denotes the *i*th member in each countably infinite sequence of objects in D.

The function d^Σ. Where α is a singular term and Σ an infinite sequence of D-objects, $d^\Sigma(\alpha)$ is that element from D that α names (if it is a name) or the *i*th element of Σ denoted by α (if α is a variable indexed by *i*).

Satisfaction.

1. If $A(\alpha_1 \ldots, \alpha_n)$ is a formula in which A is n-any predicate and $\alpha_1 \ldots, \alpha_n$ are singular terms then Σ satisfies Φ iff $\langle d^\Sigma (a_1), \ldots, d^\Sigma(\alpha_n)\rangle$ is a sequence that is a member of the set of n-triples from D denoted by the n-ary predicate A.

2. Σ satisfies ~A iff Σ does not satisfy A.

3. Σ satisfies A ∨ B iff Σ satisfies A or Σ satisfies B or both.

4. If ∀α(A) is a formula whose variable α carries the index *i*, then Σ satisfies ∀α(A) iff every countably infinite sequence of the objects in D that differs from Σ in at most its *i*th element is a sequence that satisfies Φ.

Additional Properties of the Semantics

1. *Satisfiability.* A is an I-satisfiable formula iff there is at least one interpretation under which at least one sequence Σ satisfies A.

2. *Simultaneous satisfiability.* A set Γ of formulas is simultaneously I-satisfiable iff there is at least one interpretation under which at least one Σ satisfies all the A_i in Γ.

3. *Truth for an interpretation.* A is true for I iff *every* Σ under I satisfies A.

4. *Falsity for an interpretation.* A is false for an interpretation I iff *no* Σ under I satisfies it.

5. *Models of a formula.* I is a *model* for A is true iff I is an interpretation under which A is T.

6. *Countermodels of a formula.* I is a *countermodel* for A iff I is an interpretation under which A is F.

7. *Models (Countermodels) for sets.* I is a model (countermodel) for a set Γ of formulas iff every A in Γ is T(F) in I.

8. *Models for systems.* A system, such as CQT, has a model iff the set of its theorems has a model.

9. *Countermodels*. Similarly.

10. *Validity.* A is a valid sentence (logical truth) iff for every interpretation it has a model.

Entailment. A formula A (or set thereof) entails a formula B iff there is no interpretation that is a *model* for the former that is *not a model* for the latter.

Rules:

Detachment. For any formulas A and B, B is entailed by {A, A ⊃ B}.

Generalization. For any formula A and variable α. A entails ∀α(A), provided α does not occur free in A.

P.S. These rules are relative to certain axioms. Here they are, for the record.

A1. $(A \supset (B \supset C)) \supset (A \supset B) \supset (A \supset C))$

A2. $A \supset (B \supset A)$

A3. $\sim\sim A \supset A$

A4. $\forall\beta(A) \supset S^{\beta}_{\alpha}(A)$, where α is free for β in A

A5. $(\forall\alpha\ (A) \supset B)) \supset ((A \supset \forall\alpha\ B))$

A Few Methatheoretical Remarks

1. CQT is *sound* and *complete.*

2. In its monadic version, validity of sentences and sequences is *decidable.* (The monadic version of CQT is just like the one described here, except that all predicates are one-place predicates). However, polyadic CQT is not decidable with respect to validity.

3. CQT is a *first order* system. Its domain D is restricted to *individuals.* Thus, CQT allows for quantification over individuals, but not over properties. A second order logic is one permitting quantification over both individuals and properties of individuals.