Philosophy 324A

Philosophy of Logic

2016

Note Twenty-two

EX FALSO

Let's return now to the proof of *ex falso*. **Please note with due care that the version that I'm about to present *displaces* (yes *displaces*) the version I gave earlier.** The reason why will become clear as we proceed.

1. *The new proof*

Let S be a statement-expressing sentence of English and let the "not" of "not-S" operate as sentence-negation. Then

1. S and not-S. (by assumption)
2. If S and not-S, then it is true that S and not-S. (Condition T)[1]
3. If it is true that S and not-S then, on T principle that if both hold true so does each, S is true.
4. If S is true then, on the principle that for any pair of sentences containing S at least one of them is true, at least one of S, X is true for arbitrary X.
5. If not-S is true then, on the negation principle, S is false and therefore by bivalence is not true.
6. If at least one of S, X is true and S is not, then, on the principle that if at least one of two particular sentences is true and this one is not, it's the other one that's true, X is true.

[1] Advanced by Tarski in "The Concept of Truth in Formalized Languages" as a condition of "material adequacy" for any theory of natural language truth. The full condition asserts that "S" is true if and only if S ("Snow is white" is true if and only if snow is white). The condition's biconditional structure provides that if S then it is true that S, which is the form in which we have line (2).

7. Since each of these steps save the first arises in a truth-preserving way from prior such lines, we have it that ours is a valid conditional proof of the statement that contradictions logically imply the negations of anything they imply.

2. *Probing the new proof*

(i) In approaching the S, X pair at line (4), it is necessary to bear in mind that we already have it independently that if “S and not-S” is true, so is S. In approaching the S, X pair at line (6), it is necessary to bear in mind that we already have it independently on this same assumption that S is not true.

(ii) In the general case in which we have it by assumption that at least one of two statements S*, X* is true and that S* is not true, we default to the conclusion that X* is the true member of the pair.

(iii) However, ours is not the general case. It is the quite particular case in which on the assumption of the proof we have it independently that, if at least one of S, X is true, one of them is S, without the necessity of the other one also being true. On the other hand, this is a case in which on that same assumption we also have it independently that if at least one of S, X is true and yet S isn’t true, then X is the one that is.

What we have here is the appearance of a standoff. At different validly derived stages of the proof we have it that S’s truth makes it the case that at least one of the pair S, X is true, and also that S’s non-truth makes it the case that X is. The question is whether under this assumption we can have it both ways. Given that at (4) at least one of S, X is true, it is so solely in virtue of S’s truth, and given that at (6), at least one of S, X is true, it is solely in virtue of S’s not being true, it seems reasonable to wonder whether at this juncture the sentence-operator “not-“ loses its *negational* potency? After all, if S is true and also not true, how can “not-S” remove S as a candidate for that in virtue of which at least one of S, X is true?

My own view of the matter is that this is a good and valuable question. Let’s turn now to three different and rival ways of answering it. (Note the pluralism here.)

3. *The dialethic answer*

Dialethism is the doctrine that some highly select few contradictions are true; for example, the Russell sentence and the Tarski liar sentence. This presents the dialethist with a challenge. How does the *contradictoriness* of the Russell and Tarski sentences comport with their *truth*? The dialethic answer is that these sentences answer to a *three valued* logic, whose truth values are classical T, classical F, and nonclassical T-and-F. In this logic the Russell and Tarski sentences take the third value. How does this matter here? It matters in the following way. Take the Russell sentence as an example. If “The Russell set is a member of itself and the Russell set is not a member of itself” is true-and-false, so is each of its conjuncts, “R ε R” and “R ∉ R”. Why? Because if they didn’t take T- and-F, they’d have to take either T or F. Why? Because every sentence of this logic takes one and only one of these three truth values. Pick a case: “R ε R” is T. Then “R ∉ R” is F. Now re-run the Russell proof under this valuation. You’ll end up with the conclusion that “R ε R” takes T and also takes F. This is not allowed in the logic in question. No sentence takes more than one truth value. Hence we’ll have to agree that both conjuncts of “R ε R” and “R ∉ R”take the same truth value, namely, the third one T- and–F.

Suppose now that we use the Russell sentence as the opening assumption of our new proof of *ex falso.* Then everything's hunkey-dorey until we get to line (6). If "R ε R" is true-and-false and "R ∉ R" is true-and-false, then at (6) "R ∉ R" can't negate the true-and-false "R ε R" of the pair of which X is the other member. So the proof fails at this point.

4. *Rejoinder*

Yes. If in *our* proof arbitrary "S and not-S" is both true-and-false, that puts paid to *ex falso* once and for all. But dialethists don't for a minute think that our arbitrary "S and not-S" has the privileged status of the Russell or Tarski contradictions. Dialethists try to be as classical as possible when dealing with assumptions that fail to qualify for special dialethic treatment. Very well, then. By dialethism's own lights, except for these special cases, "S and not-S" is to be treated classically. The trouble is that, when we get to line (6), some critics decide to play the *dialethic* card, thus changing horses in midstream.

Still, the question remains. How can S be true on an assumption and yet also be false on it without neutering "not" at the crisis-centre of line (6)?

5. *The cancellationist answer*

Cancellationism is an old idea. There were significant stirrings of it in the rich literatures of mediaeval logic. There is no need to go into details here. Suffice it to say that the key idea is that if someone forms the conjunction of a statement S and its negation not-S, then whatever one conjunct says is *erased* by what the other conjunct says. Here is the idea in simplified terms: If Sally tells you that 2 is the only even prime number and that it is by no means the only even prime number, she has told you nothing.

If so, that would cripple the proof. Not just at line (6), but at line (1).

6. *Rejoinder*

There is a lot that could (and deserves) to be said here. But, because time presses, let's cut to the chase. If we were to apply the cancellation option to *reductio* proofs and proofs by contradiction, we'd do a good deal of serious damage to work-a-day mathematics.

7. *John's answer* (not a head-office one)

The key question is whether at (6) "not-S" can exclude S from the choice-space between S and X for the truth that lies within, in light of the fact that we already have it that S itself verifies the assumption that at least one of S, X is true. The nub of this question – the deep centre of it – is this. At what point of the proof does "not-" lose its negational potency? If it loses its power at line (1), we'll be landed in the cancellationist camp, and will thereby have dealt a nasty blow to mathematics. That's a good reason for thinking that "not-"'s negational authority is untrifled with at line (1).

Very well, then, suppose that the "not-" of "not-S" has full negational potency with regard to S. If it *lacked* this feature at line (1), we'd lose all interest in it. From which I conclude that (1)'s interest is wholly centred on "not-"'s negational powers. The question that now presses in why would "not-" *lose* its negational potency lower down the proof's chain of truth-preserving reasoning? The fact that at line (6) it verifies *ex falso* strikes me as no reason at all to think that the S of line (1) doesn't negate the "not-S" of the same line, or that lower down the proof goes off the negational track.

The virtue of *ex falso* is that it retains the negational interest of line (1) all the way down the proof. From which I conclude that line (6) is validly derived from the proof's assumption.

7. *Final remarks*

I agree that (6) raises the question of negational potency, and that sensible people might be inclined to think that in its journey from line (1) to line (6), "not-" runs out of negational gas. I further agree that, in thinking so, it would be natural for doubters to look at less contentious places in the proof at which it might plausibly have gone wrong. Here are some of them.

- When a conjunction is made up of contradictory conjuncts, the conjunction decomposition rule is suspect. If "S and not-S" is under a cloud why would its conjuncts not be?

- When a conjunct of such a contradiction is in fact detached, its contradicted nature raises questions about the reliability of the "disjunction" law. If "S" is under a cloud, how could it not be so that "At least one of S and X is true" not also be?

- At the beginning of this note I said that a bit later on I'd explain why I decided to reformat the proof. In its present form, the proof centres on the powers of the negation-operator, whose role in life is to flip truth values. If S is true then not-S is false. If S is false, then not-S is true. Giving the proof this focus helps us see that what's really on the line here is whether "not-" retains its truth-value flipping powers under the assumption of a contradictory conjunction. In earlier versions, both the one introduced in class and the one on p. 10 of note #21, there is no mention of truth values, hence no occasion to consider whether "not-" always flips them. This omission helps disguise the fact that flipping is the principal issue of the proof.

8. *Last words*

Apart from a note that might (or might not) be posted about the final exam, this is my last post for this course. Indeed it is the last post for my final course on logic. I have been teaching logic continually since 1961. I have vivid memories of the class of that year, and will retain vivid memories of you, my last logic students, with heartfelt thanks in each case.