Philosophy 324A
Philosophy of Logic
2016

Note Nine

JUSTIFYING FORMAL REPRESENTABILITY CLAIMS

ODDS AND ENDS

1. Before we begin, let me correct an error I made in Thursday's class (Sept. 29th). We were explaining some harsh restrictions that classical logic places on canonical English, for example, that its sentences can't contain adverbs. This is true, but I went on to misspeak in saying that all verbs would be banished save for the "is" of predication. And I cited action-verbs as an example. What I intended saying is that sentences like "Bill's action in punching Henry was reprehensible", which, aside from its tensedness, purports to name an action, whereas in classical logic only *individuals* have names. Similarly for "All of Bill's actions are reprehensible", which quantifies over non-individuals. The same constraints fall on event-sentences such as "All the events in Aleppo last week were deeply troubling". It quantifies over events, and it contains an adverb ("deeply") and temporal adjective construction ("last"). While I'm at it, it would be more correct to say that sentence-operators are banned except for the operator associated with the connectives – e.g. "it is not the case that". Sorry for the confusion.

2. We can safely presume that whenever a philosopher seeks a formal semantics for the treatment of some matter of philosophical interest, she is seeking what Tarski produced in "The concept of truth in formalized languages" and "The semantic conception of truth." "And what was that?" you ask. It is this: she is trying to provide for the concept of interest a rational reconstruction; and she is doing that in precisely the way laid out in Note #8.

3. Right at the end of yesterday, I made brief mention of the Löwenheim-Skolem theorem. Here is a faster description of what it says. (**Please note that this material is not examinable**) The basic theorem was first proved in 1915 by Leopold Löwenheim (1878-1941) and proved again in 1919 by Thoraf Skolem (1887-1963), each showing that any first-order logic with identity has a countable model if it has any model at all. In 1928 Tarski showed that every such theory that has an infinite model has a model of *every* infinite cardinality. What Löwenheim and Skolem showed was that some theories – e.g. set theory and real number theory (= "analysis") – have unexpectedly *small* models. This has come to be known as the *downward* Löwenheim-Skolem theorem. What Tarski showed is that theories with infinite models have unexpectedly large models. This is now known as the *upward* Löwenheim-Skolem theorem.

PEANO ARITHMETIC

In 1989 in *Arithmetices principia*, *nova method exposita*, Guiseppe Peano did what Richard Dedekind had done the year before, but not quite as clearly. Peano introduced three primitive arithmetical notions – i.e. notions needing no definition. They are

- Zero

- (natural) number
- (immediate) successor.

He went on to formulate five principles, often called Peano's axioms:

1. Zero is a number.
2. The successor of any number is a number.
3. Zero is not the successor of any number.
4. Any two numbers with the same successor are the same number.
5. Any property of zero that is also a property of the successor of any number having it is a property of all numbers.

Peano arithmetic is the deductive closure of these arithmetics.[1]

Since (5) quantifies over properties, Peano arithmetic is a second-order system. To get a first order one, axiom (5) is replaced by an axiom schema

- For any formula A in which x does not occur bound, $(A(0) \,\&\, \forall x(A(x)) \supset A(Sx))) \supset \forall x A(x)$.

*(*The material of this section should be scanned, but isn't examinable**)

FORMAL REPRESENTATION

What makes it plausible to say that the model theory of first-order classical logic formally represents semantic properties of English, properties such as truth, logical truth, logical consequence (i.e. entailment), logical consistency, validity (i.e. truth-preservation), reference, naming, quantification etc.? The answer that's usually given is that for each of these natural language properties there is a matching formal property of the model theory established by a one-to-one correspondence which is truth-representation preserving; that is to say, for every true natural language sentence S there is some unique formal sentence A which has a model in the interpretation mandated by the model theory. Of course, this has no chance of being true as stated. What is meant is that certain select *fragments* of English are formally representable, and they are so only after being put into canonical notation. As far as I know, there has been no suitably general and principled means of specifying the maximal size of allowable fragments or of determining the correct regimentation rules whereby the chunk is put into canonical notation. Still, no one seriously doubts the formalization claim, and there is virtually no discussion of whether it's been shown to be true.

FORMAL ARITHMETIC

We now move to an example in which formal representability is front and centre and very much the subject of investigations into its *bona fides*. But first we'll have to say a little something about *formal* arithmetic. Enthusiasm for syntax both were growing and of increasing influence in the philosophy of mathematics and metamathematics, notwithstanding the Tarskian

[1] This, too, is not right as stated. The idea is that the closure of the axioms will contain all and only the true sentences of number theory. But, given that deduction is truth-preserving and the logic in question has the v-introduction rule, the PA closure will include infinitely many disjunctive truths, none of which will be a truth of arithmetic (unless all the disjuncts are themselves truths of arithmetic).

enrichment of model theory. The proof theoretic approach to mathematics was kick-started by Hilbert at the turn of the century, and advances were made by Bernays, Brouwer and numerous other talented people in their descendent chains. In a sense, proof theory and model theory were in a space-race, a race for the theoretical dominance of logical space. Of course, more moderate thinkers derived some solace from completeness and soundness proofs, thanks to which whatever we say truth-theoretically we can say without relevant loss proof-theoretically (and *vice-versa*, of course). For more doctrinaire combatants, the war was still on. Even though Peano arithmetic is now nicely axiomatized, every one of its theorems is written in a natural and meaningful language, and every one of them satisfies the natural-language truth-predicate. Some of these combatants clung to the view that such philosophically troubling notions – of meaning and of truth – have no intellectually justifiable business in rigorous mathematics. Accordingly, the challenge went out to make Peano arithmetic redundant and replace it by a thoroughly cleaned up purely formalized version.

For our purposes here, it is necessary to go into the details of how this formalization was achieved, beyond noting that it was brought about in the same *general* sort of way that classical model theory formally represents counterpart semantic properties of natural language. Once achieved, the formally achieved representation of PA was FA, the theory that formally represents Peano arithmetic. At this point, you might quite naturally ask, "Well then, what has all this to do with the issue of how formal representability claims are grounded?" To answer, we'll now have to turn to Gödel.

4. *Gödel's incompleteness proof*

In 1931 Kurt Gödel (1906-1978) proved the most momentous theorem to date in the metalogic of modern mathematical logic.[2] To achieve this result, Gödel wanted to formalize PA in a manner that would meet with Hilbertian approval. The reason why is that his intention was to spike the guns of Hilbert's incursion into logic. He would show, using methods that were Hilbertian kosher that there exists a provably unprovable sentence of FA which says of itself that it is unprovable, hence is a true but unprovable sentence of formal arithmetic. Call this sentence the Gödel sentence G. Of course, it is not literally true that G says of itself that it is unprovable. But using very clever devices – in some ways suggested of the kinds of methods employed by Cantor to generate the Liar paradox without the need of self-referential sentences. The principal device is called *Gödel-numbering.* Here, too, there is neither need nor time to go into the Gödel-numbering technical complexities of bringing G into the language of FA. The point of mentioning here has everything to do with the necessity of having *proofs* of formal representability at one's beck and call.

FORMAL REPRESENTABILITY PROOFS

Gödel's incompleteness proof would collapse were it not provably true that all primitive recursive functions are representable in FA; similarly for all primitive recursive relations. In consequence, every primitive recursive truth of PA has a corresponding theorem in FA, and every primitive recursive falsehood of PA has a negation which has a corresponding FA-theorem. Later on, it was shown that this formal representability proof generalizes to all

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[2] Kurt Gödel, "On formally undecidable propositions of *Principia Mathematica* and related systems I", in Jean van Heijenoort, editor, *From Frege to Gödel: A Reference Book in Mathematical Logic, 1879-1931,* pages 596-617, Cambridge, MA: Harvard University Press, 1967. Originally appeared in German in 1931.

recursive functions. This stronger result is not needed for the Gödel proof of incompleteness. It is powerful enough to have helped launch the modern mathematics of computable functions.

Readers unfamiliar with this mathematical jargon needn't worry. The point to grasp here is that this momentous proof of incompleteness would have crashed and burned without this proof of formal representability.

THE MORAL

The frequency of formal representability attributions in philosophy and the social and life sciences greatly outpaces demonstrations of their existence. Fortunately, there are some notable exceptions.

- Hartry Field essayed an impressive go at nominalizing thermodynamics. At the heart of his project was the formal representation of physics' qualitative properties by strictly numerical ones. He grounded this claim in a proof fashioned from the mathematics of measurement theory.

- In preservationist versions of paraconsistent logic, a number of key formal representability theorems help keep the logic in credible business. (We'll come back to paraconsistency a bit later in the course.)

Further details can be got from "Does changing the subject from A to B …? The take-away here is simply stated:

- The more deeply committed a theory is to the methodology of formal representability, the more the lack of formal representability proofs exposes the theory to hopeful whistling in the dark. (The shorter way of saying this is if you can't afford the methods, you use them at your peril.)

This concludes the first of this course's three modules. **You needn't worry over much if you're unfamiliar with some of the technical terminology of this note. The things to fasten onto securely are the last three bulleted passages on this page.**